Dual $\mathcal{PT}$-Symmetric Quantum Field Theories

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Some quantum field theories described by non-Hermitian Hamiltonians are investigated. It is shown that for the case of a free fermion field theory with a γ_5 mass term the Hamiltonian is $\mathcal{PT}$-symmetric. Depending on the mass parameter this symmetry may be either broken or unbroken. When the $\mathcal{PT}$ symmetry is unbroken, the spectrum of the quantum field theory is real. For the $\mathcal{PT}$-symmetric version of the massive Thirring model in two-dimensional space-time, which is dual to the $\mathcal{PT}$-symmetric scalar Sine-Gordon model, an exact construction of the $\mathcal{C}$ operator is given. It is shown that the $\mathcal{PT}$-symmetric massive Thirring and Sine-Gordon models are equivalent to the conventional Hermitian massive Thirring and Sine-Gordon models with appropriately shifted masses.

I. INTRODUCTION

Hermiticity is a convenient symmetry of a quantum-mechanical Hamiltonian H because it guarantees that the energy eigenvalues of H are real. (We use the term *Hermiticity* here in the usual Dirac sense, where Dirac conjugation $H^\dagger$ represents the combined transpose and complex conjugate of H.) However, there are also non-Hermitian Hamiltonians whose eigenvalues are real. A recently discovered class of such Hamiltonians is [1]

$$H = p^2 + x^2(ix)^\epsilon \quad (\epsilon > 0). \tag{1}$$

The eigenvalues of H in (1) are entirely real because H is invariant under space-time reflection and this reflection symmetry, known as $\mathcal{PT}$ *symmetry*, is unbroken [2, 3]. We say that a Hamiltonian H is $\mathcal{PT}$-symmetric if H commutes with the $\mathcal{PT}$ operator, where $\mathcal{P}$ represents parity reflection and $\mathcal{T}$ represents time reversal.

It is necessary that the eigenvalues of a Hamiltonian H be real in order for H to describe a physical theory of quantum mechanics, but the reality of the spectrum of H is not sufficient. One must also establish that the time evolution operator $U = e^{iHt}$ is unitary (norm-preserving). If H is non-Hermitian, then U is not unitary with respect to the standard Hilbert space inner product definition $\langle A|B \rangle = A^\dagger \cdot B$. However, Ref. [2] demonstrated that there is a new positive inner product $\langle A|B \rangle = A^{\mathcal{CPT}} \cdot B$ with respect to which the time evolution operator U is unitary. The operator $\mathcal{C}$ is linear and is defined by the following

system of three algebraic equations:

$$\mathcal{C}^2 = 1, \tag{2}$$
$$[\mathcal{C}, \mathcal{PT}] = 0, \tag{3}$$
$$[\mathcal{C}, H] = 0. \tag{4}$$

By solving these three simultaneous equations for the operator $\mathcal{C}$, one obtains an inner product with respect to which H is self-adjoint.

In Refs. [4, 5] it is shown that the $\mathcal{C}$ operator has the general form

$$\mathcal{C} = e^Q \mathcal{P}, \tag{5}$$

where Q is a Hermitian operator. This result makes contact with the general framework of pseudo-Hermitian Hamiltonians [6]. In Ref. [6] it is shown that the square root of the positive operator $\eta \equiv e^{-Q}$ can be used to construct a Hermitian Hamiltonian h that corresponds to the non-Hermitian Hamiltonian H. The Hamiltonians h and H are related by the similarity transformation

$$h = e^{-Q/2} H e^{Q/2}. \tag{6}$$

Perturbative methods were adopted in Refs. [4, 5, 7] for calculating the $\mathcal{C}$ operator in (5) for the case of the cubic $\mathcal{PT}$-symmetric quantum-mechanical Hamiltonian

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}x^2 + i\varepsilon x^3 \tag{7}$$

and for the cubic $\mathcal{PT}$-symmetric field-theoretic Hamiltonian density

$$\mathcal{H} = \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}m^2\varphi^2 + i\varepsilon\varphi^3. \tag{8}$$

In addition, the $\mathcal{C}$ operator has been obtained perturbatively for the quantum-field-theoretic Lee model [8] and for a $\mathcal{PT}$-symmetric version of quantum electrodynamics [9]. Semiclassical methods have also been used to construct $\mathcal{C}$ [10].

It is explained in Refs. [4, 5] how to calculate the $\mathcal{C}$ operator using perturbative methods. To summarize the procedure, we write the Hamiltonian H in the form

$$H = H_0 + \varepsilon H_1, \tag{9}$$

where H_0 and H_1 are respectively even and odd under parity reflection. We then represent the Q operator as a formal series in powers of ε:

$$Q = \varepsilon Q_1 + \varepsilon^3 Q_3 + \varepsilon^5 Q_5 + \cdots . \tag{10}$$

It has been shown that only odd powers of ε appear in this series [5]. Substituting (9) and (10) into (4) and collecting powers of ε, we obtain a sequence of algebraic equations that can be solved successively to determine the coefficients Q_1, Q_3, Q_5, and so on. The first three of these equations are

$$
\begin{aligned}
{}[Q_1, H_0] &= 2H_1, \\
[Q_3, H_0] &= \tfrac{1}{6}[Q_1, [Q_1, H_1]], \\
[Q_5, H_0] &= -\tfrac{1}{360}[Q_1, [Q_1, [Q_1, [Q_1, H_1]]]] + \tfrac{1}{6}[Q_1, [Q_3, H_1]] + \tfrac{1}{6}[Q_3, [Q_1, H_1]].
\end{aligned}
\tag{11}
$$

In principle, the system of equations (11) can be solved for the perturbation coefficients Q_1, Q_3, Q_5, $\cdots$ to any finite order. However, for the cubic quantum theories that have been studied using this procedure, it has not so far been possible to sum the perturbation series in (10) to all orders.

For the field-theoretic models considered in the present paper, we are able to calculate the perturbation coefficients Q_1, Q_3, Q_5, $\cdots$ to all orders and subsequently to sum the series (10) *exactly*. Using this procedure we can construct the Q operator in closed form for a $\mathcal{PT}$-symmetric version of the massive Thirring model in $(1+1)$-dimensional space-time. This procedure can be formally extended to the Thirring model in $(3+1)$ dimensions, even though this model is not renormalizable in higher dimensions. The special feature of $(1+1)$ dimensions is that bosonization prescriptions exist that permit us to construct bosonic theories dual to these fermionic theories. In the present case the massive Thirring model is dual to the purely bosonic Sine-Gordon model. It then follows that Q is immediately calculable for the dual bosonic theory.

This paper is organized as follows: In Sec. II we examine a $\mathcal{PT}$-symmetric free fermionic field theory with a γ_5 mass term and show that there is a region of unbroken $\mathcal{PT}$ symmetry for which the energy levels are real and a region of broken $\mathcal{PT}$ symmetry for which the energy levels are complex. We calculate the exact Q operator for this theory. In Sec. III we study the $\mathcal{PT}$-symmetric version of the massive Thirring model in $(1+1)$-dimensional space-time and in Sec. IV we study the equivalent bosonic $\mathcal{PT}$-symmetric Sine-Gordon model in $(1+1)$-dimensional space-time. For these models we calculate the Q operator exactly and in closed form and identify the corresponding Hermitian Hamiltonians.

II. $\mathcal{PT}$-SYMMETRIC FREE FERMION THEORY WITH A γ_5 MASS TERM

The Lagrangian density for a conventional Hermitian free fermion field theory is

$$\mathcal{L}(\mathbf{x}, t) = \bar{\psi}(\mathbf{x}, t)(i\slashed{\partial} - m)\psi(\mathbf{x}, t) \tag{12}$$

and the corresponding Hamiltonian density is

$$\mathcal{H}(\mathbf{x}, t) = \bar{\psi}(\mathbf{x}, t)(-i\vec{\nabla} + m)\psi(\mathbf{x}, t), \tag{13}$$

where $\bar{\psi}(\mathbf{x}, t) = \psi^\dagger(\mathbf{x}, t)\gamma_0$.

A.   Free Fermion Theories in Two-Dimensional Space-Time

We consider first the case of two-dimensional space-time. In $(1+1)$-dimensional space-time we adopt the conventions used in Ref. [12]:

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{14}$$

With these definitions we have $\gamma_0^2 = 1$ and $\gamma_1^2 = -1$. We also define

$$\gamma_5 = \gamma_0\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{15}$$

so that $\gamma_5^2 = 1$.

In $(1+1)$ dimensions the parity-reflection operator $\mathcal{P}$ has the effect

$$\begin{aligned}
\mathcal{P}\psi(x,t)\mathcal{P} &= \gamma_0\psi(-x,t), \\
\mathcal{P}\bar{\psi}(x,t)\mathcal{P} &= \bar{\psi}(-x,t)\gamma_0.
\end{aligned} \tag{16}$$

The effect of the time-reversal operator $\mathcal{T}$ is similar to that of the parity operator,

$$\begin{aligned}
\mathcal{T}\psi(x,t)\mathcal{T} &= \gamma_0\psi(x,-t), \\
\mathcal{T}\bar{\psi}(x,t)\mathcal{T} &= \bar{\psi}(x,-t)\gamma_0,
\end{aligned} \tag{17}$$

except that $\mathcal{T}$ is *anti*-linear and therefore takes the complex conjugate of complex numbers.

It is easy to see that the Hamiltonian $H = \int dx\,\mathcal{H}(x,t)$, where $\mathcal{H}$ is given in (13), is Hermitian: $H = H^\dagger$. Also, H is separately invariant under parity reflection and under time reversal:

$$\mathcal{P}H\mathcal{P} = H \quad \text{and} \quad \mathcal{T}H\mathcal{T} = H.$$

Now let us construct a *non*-Hermitian Hamiltonian by adding a γ_5-dependent mass term to the Hamiltonian density in (13):

$$\mathcal{H}(x,t) = \bar{\psi}(x,t)(-i\overset{\leftrightarrow}{\nabla} + m_1 + m_2\gamma_5)\psi(x,t) \quad (m_2 \text{ real}). \tag{18}$$

The Hamiltonian $H = \int dx\,\mathcal{H}(x,t)$ associated with this Hamiltonian density is not Hermitian because the m_2 term changes sign under Hermitian conjugation. This sign change occurs because γ_0 and γ_5 anticommute. Also, H is not invariant under $\mathcal{P}$ or under $\mathcal{T}$ separately because the m_2 term changes sign under each of these reflections. However, H *is* invariant under combined $\mathcal{P}$ and $\mathcal{T}$ reflection. Thus, H is $\mathcal{PT}$-symmetric: $H^{\mathcal{PT}} = \mathcal{PT}H\mathcal{PT} = H$. The field equation associated with $\mathcal{H}$ in (18) is

$$\left(i\overset{\leftrightarrow}{\partial} - m_1 - m_2\gamma_5\right)\psi(x,t) = 0. \tag{19}$$

If we iterate this equation and use $\overset{\leftrightarrow}{\partial}^2 = \partial^2$, we obtain the two-dimensional Klein-Gordon equation

$$\left(\partial^2 + \mu^2\right)\psi(x,t) = 0, \tag{20}$$

where $\mu^2 = m_1^2 - m_2^2$. Thus, the physical mass that propagates under this equation is real when the inequality

$$m_1^2 \geq m_2^2 \tag{21}$$

is satisfied. This condition defines the two-dimensional parametric region of *unbroken* $\mathcal{PT}$ symmetry. When (21) is not satisfied, the $\mathcal{PT}$ symmetry is *broken*. Figure 1 displays the regions of broken and unbroken $\mathcal{PT}$ symmetry. Note that the special case of Hermiticity is restricted to a one-dimensional region (the line $m_2 = 0$).

We will now show how to calculate the $\mathcal{C}$ operator associated with the $\mathcal{PT}$-symmetric Hamiltonian density $\mathcal{H}$ in (18). We begin by letting $m_1 = m$ and $m_2 = \varepsilon m$ and rewriting $\mathcal{H}$ in the form

$$\mathcal{H}(x,t) = \bar{\psi}(x,t)(-i\overset{\leftrightarrow}{\nabla} + m + \varepsilon m\gamma_5)\psi(x,t). \tag{22}$$

We treat ε as a perturbation parameter and decompose the Hamiltonian associated with $\mathcal{H}$ in (22) as in (9): $H_0 = \iint dx\,dy\,\bar{\psi}(x,t)D(x,y)\psi(y,t)$, where $D(x,y) = (-i\overset{\leftrightarrow}{\nabla} + m)\delta(x-y)$ and $H_1 = \int dx\,\bar{\psi}(x,t)m\gamma_5\psi(x,t)$.

We seek the $\mathcal{C}$ operator in the form (5). Note that because $\mathcal{C}$ commutes with the Hamiltonian, it is time independent. Thus, we may calculate the Q operator at any time. For the Hermitian Hamiltonian H_0 the Q operator vanishes. However, on the basis of the work done in Ref. [4] we expect that when $\varepsilon \neq 0$, Q takes the form of the perturbation expansion in (10). We assume that the Q_n operator is bilinear in the fields:

$$Q_n = \iint dx\, dy\, \bar{\psi}(x,t) G_n(x,y) \psi(y,t),\tag{23}$$

where $G_n(x,y)$ are functions to be determined. Then, the first of the commutation relations in (11) reads

$$\iiiint dx\, dy\, dz\, dw\, [\bar{\psi}(x,t) G_1(x,y)\psi(y,t), \bar{\psi}(z,t) D(z,w)\psi(w,t)]$$
$$= 2m \int dx\, \bar{\psi}(x,t)\gamma_5\psi(x,t).\tag{24}$$

The canonical equal-time anticommutation relations $\{\psi^\dagger(x,t), \psi(y,t)\} = \delta(x-y)$ may be used to simplify the left side of this equation:

$$-\iint dx\, dy\, \bar{\psi}(x,t) \left(m[\gamma_0, G_1(x,y)] + i\{\gamma_0 \vec{\nabla}, G_1(x,y)\} \right) \psi(y,t)$$
$$= 2m \int dx\, \bar{\psi}(x,t)\gamma_5\psi(x,t).\tag{25}$$

This equation does not determine $G_1(x,y)$ uniquely, but $G_1(x,y) = -\gamma_1\delta(x-y)$ is a particular solution [11]. Inserting this solution into the second equation of (11), we find that the right side reduces to $(2/3)H_1$, giving $G_3 = G_1/3$. The third equation of (11) gives $G_5 = G_1/5$.
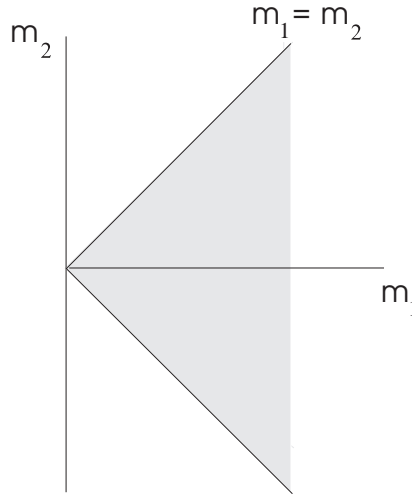

FIG. 1: Parametric regions of broken and unbroken $\mathcal{PT}$ symmetry for the Hamiltonian H in (18) in the (m_1, m_2) plane. The region of unbroken $\mathcal{PT}$ symmetry $m_1^2 \geq m_2^2$ is shaded. For these values of the parameters m_1 and m_2, the Dirac equation describes the propagation of particles having real mass. The special case of Hermiticity is obtained on the line $m_2 = 0$, which lies at the center of the region of unbroken $\mathcal{PT}$ symmetry. The region of broken $\mathcal{PT}$ symmetry $m_1^2 < m_2^2$ is unshaded.

This process successively generates a hyperbolic arc tangent, with the all-orders result

$$Q = -\tanh^{-1}\varepsilon \int dx\, \bar{\psi}(x,t)\gamma_1\psi(x,t) = -\tanh^{-1}\varepsilon \int dx\, \psi^\dagger(x,t)\gamma_5\psi(x,t). \qquad (26)$$

(The inverse hyperbolic tangent function in this equation requires that $|\varepsilon| \leq 1$, or equivalently $m_1^2 \geq m_2^2$, which corresponds to the shaded region of unbroken $\mathcal{PT}$ symmetry in Fig. 1.) We can verify (26) by constructing

$$h = \exp\left[\tfrac{1}{2}\tanh^{-1}\varepsilon \int dx\, \psi^\dagger(x,t)\gamma_5\psi(x,t)\right] H \exp\left[-\tfrac{1}{2}\tanh^{-1}\varepsilon \int dx\, \psi^\dagger(x,t)\gamma_5\psi(x,t)\right]. \qquad (27)$$

By virtue of the Lorentz-like commutation relations

$$[\gamma_5, \gamma_0] = -2\gamma_1, \qquad [\gamma_5, \gamma_1] = -2\gamma_0, \qquad (28)$$

h in (27) reduces to

$$h = \int dx\, \bar{\psi}(x,t)(-i\overleftrightarrow{\nabla} + \mu)\psi(x,t), \qquad (29)$$

where

$$\mu^2 = m^2(1-\varepsilon^2) = m_1^2 - m_2^2, \qquad (30)$$

in agreement with (20). The only effect in going from H to h is to change the γ_5-dependent mass term $m\bar{\psi}(1+\varepsilon\gamma_5)\psi$ to a normal mass term $\mu\bar{\psi}\psi$. We conclude that the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian density in (18) is equivalent to the Hermitian Hamiltonian density in (13) with m replaced by μ.

B. Free Fermion Theories in Four-Dimensional Space-Time

In $(3+1)$ dimensions the analogs of (16) and (17) are

$$\begin{aligned}
\mathcal{P}\psi(\mathbf{x},t)\mathcal{P} &= \gamma_0\psi(-\mathbf{x},t), \\
\mathcal{P}\bar{\psi}(\mathbf{x},t)\mathcal{P} &= \bar{\psi}(-\mathbf{x},t)\gamma_0,
\end{aligned} \qquad (31)$$

and

$$\begin{aligned}
\mathcal{T}\psi(\mathbf{x},t)\mathcal{T} &= \mathbf{C}^{-1}\gamma_5\psi(\mathbf{x},-t), \\
\mathcal{T}\bar{\psi}(\mathbf{x},t)\mathcal{T} &= \bar{\psi}(\mathbf{x},-t)\gamma_5\mathbf{C},
\end{aligned} \qquad (32)$$

where $\mathbf{C}$ is the charge-conjugation matrix, defined by $\mathbf{C}^{-1}\gamma_\mu\mathbf{C} = -\gamma_\mu^{\mathrm{T}}$. In $(1+1)$ dimensions $\mathbf{C}$ reduces to γ_0, as in Eq. (17).

The resulting Q is given by

$$Q = -\tanh^{-1}\varepsilon \int d\mathbf{x}\, \psi^\dagger(\mathbf{x},t)\gamma_5\psi(\mathbf{x},t), \qquad (33)$$

which is the three-dimensional generalization of the right side of (26). The validity of (33) is due to the commutation relations

$$[\gamma_5, \gamma_0] = -2\gamma_0\gamma_5 \quad [\gamma_5, \gamma_0\gamma_5] = -2\gamma_0, \qquad (34)$$

which are the generalizations of (28). Moreover, the invariance of the kinetic term is assured by the commutation relation $[\gamma_5, \gamma_0\gamma_\mu] = 0$.

In $(3+1)$ dimensions the Hermitian Hamiltonian h, which is equivalent to the non-Hermitian Hamiltonian H with a γ_5 mass term, again has the shifted mass μ in (30).

III. $\mathcal{PT}$-SYMMETRIC MASSIVE THIRRING MODEL

Our starting point, again in (1+1) dimensions, is the Lagrangian density for the conventional massive Thirring model

$$\mathcal{L} = \bar{\psi}(i\slashed{\partial} - m)\psi + \tfrac{1}{2}g(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi), \tag{35}$$

with the corresponding Hamiltonian density

$$\mathcal{H} = \bar{\psi}(-i\slashed{\nabla} + m)\psi - \tfrac{1}{2}g(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi), \tag{36}$$

This model is known [12] to be equivalent to the Sine-Gordon model (see Sec. IV) with the correspondence

$$\frac{\lambda^2}{4\pi} = \frac{1}{1 - g/\pi}, \tag{37}$$

so that, in particular, the free fermion theory is equivalent to the Sine-Gordon model with the special value for the coupling constant $\lambda^2 = 4\pi$.

The modification analogous to that of Eq. (22) is the introduction of a γ_5-dependent mass according to

$$\mathcal{H} = \bar{\psi}(-i\slashed{\nabla} + m + \varepsilon m\gamma_5)\psi - \tfrac{1}{2}g(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi), \tag{38}$$

The additional term is non-Hermitian but $\mathcal{PT}$-symmetric because it is odd under both parity reflection and time reversal.

In Sec. II we considered the case $g = 0$. The Q operator for the interacting case $g \neq 0$ is in fact identical to the Q operator for the case $g = 0$ because in $(1+1)$-dimensional space the interaction term $(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)$ commutes with the Q in (26). Thus, once again we conclude that the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian density in (38) is equivalent to the Hermitian Hamiltonian density in (36) with the mass m replaced by μ in (30).

The same holds true for the $(3 + 1)$-dimensional interacting Thirring model by virtue of the commutation relation $[\gamma_5, \gamma_0\gamma_\mu] = 0$, but because this higher-dimensional field theory is nonrenormalizable, the Q operator may only have a formal significance.

IV. $\mathcal{PT}$-SYMMETRIC SINE-GORDON MODEL

The massive Thirring Model (35) is dual to the conventional Sine-Gordon model in (1+1) dimensions whose Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\varphi)^2 + \tfrac{m^2}{\lambda^2}(\cos\lambda\varphi - 1), \tag{39}$$

and whose corresponding Hamiltonian density is

$$\mathcal{H} = \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{m^2}{\lambda^2}(1 - \cos\lambda\varphi), \tag{40}$$

where $\pi(x,t) = \partial_0\varphi(x,t)$, and in $(1+1)$-dimensional space $\nabla\varphi(x,t)$ is just $\partial_1\varphi(x,t)$.

The $\mathcal{PT}$-symmetric extension (38) of the modified Thirring model is, by the same analysis, dual to a modified Sine-Gordon model with Hamiltonian density

$$\mathcal{H} = \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{m^2}{\lambda^2}(1 - \cos\lambda\varphi - i\varepsilon\sin\lambda\varphi), \tag{41}$$

which is PT-symmetric but no longer Hermitian.

On the basis of duality, the form of the operator Q_1 is easy to guess:

$$Q_1 = \xi_1 \int dx\, \pi(x,t), \tag{42}$$

where ξ_1 is a constant. The commutator of Q_1 with H_0 is

$$
\begin{aligned}
[Q_1, H_0] &= \xi_1 \iint dx\, dy\, \left[\pi(y,t), \tfrac{1}{2}[\nabla\varphi(x,t)]^2 - \tfrac{m^2}{\lambda^2}\cos[\lambda\varphi(x,t)] \right] \\
&= i\xi_1 \int dx\, \left(\nabla^2\varphi(x,t) - \frac{m^2}{\lambda}\sin[\lambda\varphi(x,t)] \right),
\end{aligned} \tag{43}
$$

by virtue of the canonical commutation relation $[\varphi(x,t), \pi(y,t)] = i\delta(x-y)$. The first term, being a total derivative, integrates to zero, and identifying the remainder with $2H_1$ gives $\xi_1 = 2/\lambda$.

The form of Q_3 is the same as that for Q_1 with ξ_1 replaced by the constant ξ_3. Substituting into the second equation of (11), we obtain $\xi_3 = \xi_1/3$. Similarly we obtain $\xi_5 = \xi_1/5$, giving a strong indication that we are again generating the series for $\tanh^{-1}\varepsilon$ and that the all-orders result for Q is

$$Q = \frac{2\tanh^{-1}\varepsilon}{\lambda} \int dx\, \pi(x,t). \tag{44}$$

We can verify this result *a posteriori* by constructing the equivalent Hermitian Hamiltonian:

$$h = \exp\left[-\frac{\tanh^{-1}\varepsilon}{\lambda} \int dx\, \pi(x,t) \right] H \exp\left[\frac{\tanh^{-1}\varepsilon}{\lambda} \int dx\, \pi(x,t) \right]. \tag{45}$$

It is interesting that the operation that transforms H to h has precisely the effect of shifting the boson field φ by an imaginary constant:

$$\varphi \to \varphi + i\frac{\tanh^{-1}\varepsilon}{\lambda}. \tag{46}$$

Under this transformation the interaction term $m^2\lambda^{-2}(1-\cos\lambda\varphi - i\varepsilon\sin\lambda\varphi)$ in (41) becomes $-m^2\lambda^{-2}(1-\varepsilon^2)\cos\lambda\varphi$, apart from an additive constant. Hence, h is the Hamiltonian for the conventional Sine-Gordon model, but with mass μ given by (30). This change in the mass is exactly the same as we observed in the fermionic theory discussed in Sec. III. Note that h, being Hermitian, is even in the parameter ε that breaks the Hermiticity of H.

The idea of generating a non-Hermitian but PT-symmetric Hamiltonian from a Hermitian Hamiltonian by shifting the field operator as in (46), first introduced in the context of quantum mechanics in Ref. [13], suggests a new approach to generating solvable fermionic PT-invariant models whenever there is a boson-fermion duality.

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